How do Expected Returns Affect Turnover?

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Abstract

I analyze a simple model of heterogeneous and risk-neutral traders, with fluctuating valuations, who trade a single asset. I show that the model predicts different signs for the empirical relationship between average expected returns and aggregate asset turnover under two apparently similar sets of assumptions. Under the first set of assumptions, I assume that market participants differ in their discount factors, but are rational. I show that this implies a positive relationship between turnover and expected returns, as a low discount factor leads to high turnover, and that the required return to hold the asset must be high. The opposite result emerges if the heterogeneity stems from agents differing in their expectations about future asset returns. In this case, a high expected return actually decreases the incentive of agents to engage in transactions. To learn about the empirical relationship of expected returns and volume, I analyze the US stock market and the US housing market, for which both model-based and survey-based expectation data are available. I find that the evidence on the relationship is in fact mixed: there is no strong relationship between turnover and expectation measures, and the results varies greatly by employed measure of expectations. If anything, I observe a moderately positive relationship in the housing market. These results seemingly provide evidence against the view that high turnover is a symptom of high return expectations ("Exuberance").

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1 Introduction

High trading volume (more sensibly analyzed relative to the overall asset stock, so in terms of turnover) is often cited as an ubiquitous feature of times of exuberant investor sentiment or even asset price "bubbles".[1] However, it is far from obvious why agents would trade more often when they are optimistic, as from a rational perspective trade should be governed by agents' valuations of the asset relative to those of other market participants, which determine the gains from trade. This is a question that lends itself to be explored more deeply in a model of heterogeneous agents. In such a model, if the average market participant or the marginal investor become more optimistic, will there be more trade and which are the crucial assumptions to deliver such a result?

In this paper, I derive the implications of a simple model with risk-neutral heterogeneous agents for the relationship of turnover and return expectations. In fact, this simple model of traders, who are heterogeneous in either their discount factors or their return expectations, predicts that the sign of the turnover-expected return relationship depends on which of the two is the underlying source of heterogeneity. A crucial insight is that if the distribution of the "source" around its mean (which can vary over time) is fixed, a low level of the "source" variable (so either low discount factors, or overall pessimist expectations) implies high turnover, as gains from trade are larger relative to the transaction costs (assumed to be proportional to the price).However, in equilibrium the average discount factor is the inverse of the expected return, following the logic of market clearing. Thus, in a world where agents differ in their discount factors, high expected returns (which imply low discount factors, as agents demand a high expected return to be compensated for holding the asset despite valuing it so little in the future) lead to high turnover. But in a world in which agents differ in their return expectations, instead of their discount factors, a high average return expectation – optimism – coincides with little trading activity.

Having derived these results, I then take a closer look at the correlational empirical relationship between asset turnover and various measures of expectations, both in the US stock and housing markets. Using both model-based expectations (such as the dividend-price ratio), which impose the assumption of rational expectations, as well as various survey-based expectation measures, I find that in fact there is no significant correlation between turnover and return expectations, neither positive, nor negative, if observed over the long-term. If anything, there is a moderately positive relationship in the housing market, although it is not conclusive.

This paper adds to a growing literature that thinks carefully about the relationship of empirically measured expectations, both model- and survey-based, and asset market phenomena. While most other papers have

[1] For a general motivation of this thinking, see Hong and Stein (2007).

focused on returns and other phenomena related to asset prices, my paper is unique in its approach as it analyzes the relationship of measures of return expectations and trading activity. There exists an established literature indirectly deriving measures of return expectations from standard models of asset markets, examples of which are Campbell and Cochrane (1999), Lettau and Ludvigson (2001), and Cochrane (2011), the last of which provides a high-level summary the literature. However, turnover does not play a role in these considerations, and neither do heterogeneous expectations. Unsurprisingly, turnover has received some attention in the asset price literature, most prominently by Lo and Wang (2006, 2004) and Lo, Mamaysky, and Wang (2004). These papers focus primarily on the how to employ turnover to derive an additional asset pricing factor and do not dwell on expectations or the sources of traders' heterogeneity.[2]

A more recent literature analyzes the relationship of survey-based expectation measures and returns (as well as the aforementioned model-based expectation measures) and finds that a.) they appear to be measuring something meaningful (not just noise) and, b.) that they are seemingly at odds with model-based measures: e.g. Greenwood and Shleifer (2014) analyze long-running expected return surveys and find that they are negatively associated with model based returns, as well as with future returns. Adam, Mateev, and Nagel (2021) explore whether participants in expectation surveys do not report their beliefs but rather their preferences, and reject this hypothesis. Relative to these papers, my paper focuses on the relationship of heterogeneous expectations and trading activity.

Finally, there exists a literature that associates the occurrence of trading frenzies with bubbly or "exuberant" episodes, the case for which is made in Gallant, Rossi, and Tauchen (1992), Scheinkman and Xiong (2003) and Hong and Stein (2007). More recently, there have been several theoretical models of bubbly episodes, in which high turnover is predicted, e.g. Barberis et al. (2018) devise a model of extrapolative bubbles, which predicts a) that there is substantial trading volume in bubbly episodes, and b) that there should be a positive relationship between past return and trading volume. Bordalo, Gennaioli, and Kwon (2020) explore a model of excessive optimism about the development of asset fundamentals and link it to asset price bubbles, which come with high turnover. This literature motivates my research question about what a simple model of heterogeneous agents can tell us about the relationship of expectations and turnover.

My paper differs from all listed papers in the fact that I derive the relationship of turnover and expectations from a simple model of heterogeneous traders, focus on the role played by different assumptions on the

[2]A related, but not immediately relevant, strand of the literature is that of Chen, Hong, and Stein (2001) who think about how turnover fares as a predictor of future returns. They show that negative skewness of returns is more pronounced in periods following heavy trading activity. This provides some evidence in favor of the idea of differences-of-opinion: when disagreement is large, there is would be more trading activity and large negative price movements are more likely, as the traders that "sit out" have more negative opinions. However, they and Greenwood, Shleiger, and You (2018) find that high turnover does not forecast expected returns, a result that appears contradictory to the differences-of-opinion literature.

source of heterogeneity and test the clear predictions against both model- and survey-based measures of expectations. The rest of the paper is structured as follows: Section 2 presents the prediction of a simple model of heterogeneous traders and time-varying expectations, Section 3 presents an empirical test of these predictions. Section 4 summarizes the results and concludes.

2 A Simple Model of Expected Returns and Turnover

2.1 A Model of Heterogeneous and Time-varying Discount Factors

I model a discrete-time economy in which a continuous unit mass of agents trade among themselves a single asset, which is in supply N. Each agent is indexed by $i \in [0,1]$ and receives a dividend from the asset while being its "owner". Agents are risk–neutral and have rational expectations, discount the future at discount factor β_t^i (which fluctuates over time and differs between agents), and have a unit demand for the asset (so they decide to either hold the asset or not). The asset issues a dividend in each period t, denoted by d_t. In the beginning of each period the dividend is paid out, then agents can decide whether they want to pay price q_t to purchase the asset in a Walrasian market. I will assume that the distribution f_β of the β_t^i is symmetric and fixed around a value β_t, which itself however is allowed to fluctuate over time. If a transaction occurs, both buyer and seller have to pay a transaction costs κ that is proportional to the price q_t. The transaction costs are symmetric between buyer and seller.[3]

We can express this model in terms of the following value functions. The value for individual i of being an owner of an asset in t is

$$V^o\left(\beta_t^i, q_t\right) = \max\{d_t + (1-\kappa)\, q_t + \beta_t^i E_t\left[V^{no}\left(\beta_{t+1}^i, q_{t+1}\right)\right], \quad d_t + \beta_t^i E_t\left[V^o\left(\beta_{t+1}^i, q_{t+1}\right)\right]\},$$

while the value of not being an owner is accordingly

$$V^{no}\left(\beta_t^i, q_t\right) = \max\{-(1+\kappa)\, q_t + \beta_t^i E_t\left[V^o\left(\beta_{t+1}^i, q_{t+1}\right)\right], \quad \beta_t^i E_t\left[V^{no}\left(\beta_{t+1}^i, q_{t+1}\right)\right]\}.$$

The model's timing is as follows: in the beginning of t, owners receive d_t, and then make their trading decisions. In equilibrium, there are thresholds for the individual discount factors separating sellers from

[3] This is merely an expositional assumption and any distribution of the transaction cost among the transaction parties would deliver equivalent insights.

non-sellers and buyers from non-buyers. I denote these thresholds as: $\bar{\beta}_t$ for the discount factor above which an agent becomes a buyer, and $\underline{\beta}_t$ for the discount factor below which an agent becomes a seller. The asset price q_t must clear the market in equilibrium:

$$(1-N)\,\bar{\mathrm{p}}_t = N\underline{\mathrm{p}}_t, \tag{1}$$

where $\bar{\mathrm{p}}_t$ denotes the share of non-owners that will buy in period t and $\underline{\mathrm{p}}_t$ denotes the share of owners that will sell in period t. For simplicity, I always assume that $N = 1/2$, which means that at any time exactly half of the agents will be owners.

Proposition 1. *Under the assumption that f_β is a symmetric distribution that is only time-varying in its first moment β_t, the individual β_t^i are drawn from this distribution in an i.i.d fashion, and that N=1/2, we have:*

1. The asset price follows a standard pricing equation

$$q_t = \beta_t E_t\left[d_{t+1} + q_{t+1}\right] \tag{2}$$

2. Turnover is determined by

$$\bar{p}_t = 1 - F_\beta\left(\frac{1+\kappa}{E_t\left[R_{t+1}\right]}\right), \tag{3}$$

where

$$R_{t+1} = \frac{d_{t+1} + q_{t+1}}{q_t}. \tag{4}$$

This proposition implies that: a.) although we have differences in the individual valuations of the asset, they have no effect on the equilibrium price, which is only determined by the average discount factor and the (expected) dividends, and b.) turnover is a positive function of expected returns. The mechanism is simple: in equilibrium the expected return is determined by the average discount factor

$$\frac{1}{\beta_t} = E_t\left[R_{t+1}\right].$$

Trading with another agent presents a more attractive venture if prices are low relative to average expected returns, as transaction costs are proportional to the asset price q_t.

This effect increases with the expected return, as the distribution of individual discount factors F_β is fixed around β_t. Concretely, we have that the threshold discount factor to buy shrinks by a factor larger than one, if the average discount factor falls[4]:

$$\bar{\beta}_t = (1 + \kappa)\, \beta_t.$$

2.2 Heterogeneous and Time-varying Discount Factors vs Subjective and Time-varying Expectations

In the version of the model that was presented above, I have employed the assumption of an individual discount factor, which is a.) time-varying (every period agents draw a new β_t^i), and b.) heterogeneous between market participants. This induces trading decisions, driven by what could either be considered fluctuations over time in the discount factor or in their required return for reasons outside the model. In the above model, agents are rational and differ in their discount factors, a set-up which maps empirically to model-based expectation measures, such as the dividend-price ratio. However, I will also empirically test the relationship of asset turnover and survey-based expectations measures. For this it is a sensible assumption that: a.) the market participants have heterogeneous return expectations, and b.) the mean of these return expectations fluctuates over time, which are both empirical phenomena that we observe in surveys on expectations.

In the following, I want to show that the model is essentially identical under this assumption, except for the crucial insight that in this version predicts the exact opposite sign for the turnover-expected return relationship. Consider a second model that only differs in its set-up in the following points:

1. While the discount factor varies over time, it is not heterogeneous across market participants. Every trader discounts at the same β_t.
2. The expectations of the future value of owning the asset relative to the current price $E_t^i\left[\left\{V^o\left(\beta_{t+1}, q_{t+1}\right) - V^{no}\left(\beta_{t+1}, q_{t+1}\right)\right\}\right.$ are heterogeneous between agents. In fact, it adheres to a distribution $f_{\frac{\varepsilon}{q}}$ around a time-varying mean, which is $E_t\left[V^o\left(\beta_{t+1}, q_{t+1}\right) - V^{no}\left(\beta_{t+1}, q_{t+1}\right)/q_t\right]$.

Note that as q_t is known to all market participants in period t

$$E_t^i\left[V^o\left(\beta_{t+1}, q_{t+1}\right) - V^{no}\left(\beta_{t+1}, q_{t+1}\right)/q_t\right] = E_t^i\left[V^o\left(\beta_{t+1}, q_{t+1}\right) - V^{no}\left(\beta_{t+1}, q_{t+1}\right)\right]/q_t.$$

Under these slightly different circumstances, there again will be thresholds:

$$\bar{E}_t\left[\left\{V^o\left(\beta_{t+1}, q_{t+1}\right) - V^{no}\left(\beta_{t+1}, q_{t+1}\right)\right\}\right]/q_t \text{ and } \underline{\mathrm{E}}_t\left[\left\{V^o\left(\beta_{t+1}, q_{t+1}\right) - V^{no}\left(\beta_{t+1}, q_{t+1}\right)\right\}\right]/q_t$$

for buying and selling.

[4] If the dispersion of β_t^i would increase for lower β_t, this would be a counteracting effect. However, it would be unclear why the dispersion of beliefs would increase, while the expected return increases. However, to analyze the distribution of disaggregated return expectations survey responses is certainly an interesting future research project.

I will use $\mathcal{E}_t^i$ and $\mathcal{E}_t$ as a shorthand for $E_t^i\left[V^o\left(\beta_{t+1}, q_{t+1}\right) - V^{no}\left(\beta_{t+1}, q_{t+1}\right)\right]$ and $E_t\left[V^o\left(\beta_{t+1}, q_{t+1}\right) - V^{no}\left(\beta_{t+1}, q_{t+1}\right)\right]$, respectively, as well as $\bar{\mathcal{E}}_t$ and $\underline{\mathcal{E}}_t$ as shorthands for the threshold values for $E_t^i\left[V^o\left(\beta_{t+1}, q_{t+1}\right) - V^{no}\left(\beta_{t+1}, q_{t+1}\right)\right]$ at which market participants buy and sell.

The following proposition summarizes the equilibrium of the model in this case.

Proposition 2. *Under the assumption that $f_{\frac{\mathcal{E}}{q}}$ is a symmetric distribution that is only time-varying in its first moment $\mathcal{E}_t/q_t$, the individual $\mathcal{E}_t^i/q_t$ are drawn from this distribution in an i.i.d fashion (across time and individual agents), and that N=1/2, we have:*

1. The asset price follows a standard pricing equation

$$q_t = \beta_t E_t\left[d_{t+1} + q_{t+1}\right] \tag{5}$$

2. Turnover is determined by

$$\bar{p}_t = 1 - F_{\frac{\mathcal{E}}{q}}\left(\left(1+\kappa\right) E_t\left[R_{t+1}\right]\right), \tag{6}$$

where

$$R_{t+1} = \frac{d_{t+1} + q_{t+1}}{q_t}. \tag{7}$$

In fact the threshold for buying is $\bar{\mathcal{E}}_t = \left(1+\kappa\right)\mathcal{E}_t$ and we have again that in equilibrium $1/\beta_t = E_t\left[R_{t+1}\right]$. The model's equilibrium behavior is identical to before, with one exception: instead of a positive relationship of turnover and expected returns this model predicts a negative relationship! The only thing that fundamentally changed between the two models was the source of heterogeneity. In the first model the source of heterogeneity is the discount factor, in the second the source of heterogeneity are dispersed expectations about the future value of the asset and thus its return. In the following, I want to explain the mechanism that drives these results and explain what causes the difference.

2.3 Interpretation - Understanding the Mechanism

In both versions of the model, that I have presented the mechanism by which heterogeneity generates trade works in a similar manner. If we ordered the market participants according to how much they valued the asset, this order would be reshuffled every period and thus generate a motive for trade. With the simplifying assumption that $N = 1/2$, which means that always half of the market participants will hold the asset, and that the heterogeneous variable changes over time in an i.i.d. fashion, I am analyzing the extreme case: the cue is reshuffled every period, without memory of the past. In a world without transaction costs ($\kappa = 0$), the half of the mass of market participants who value the asset the most would always hold the asset and trade every period would be constant: those who move to the top half buy the asset from those who move down. In the language of the model: without transaction costs the thresholds to both buy and sell are just identical to the mean of the respective distribution: $\beta_t = \bar{\beta}_t = \underline{\beta}_t$ and $\mathcal{E}_t = \bar{\mathcal{E}}_t = \underline{\mathcal{E}}_t$. Transaction costs introduce a wedge between the buying and selling thresholds and create a no-trade region, in which market participants do not buy or sell, because the transaction costs are higher than the gains of trade. In both models the average discount factor and the average expected return vary over time[5], in fact in both models:

$$\beta_t = \frac{1}{E_t\left[R_{t+1}\right]}. \tag{8}$$

This equation tells us that the average discount factor must equal the inverse of the expected market return, as agents are risk-neutral. This means that the price will be such that the return equals to the average discount factor. Would it be higher, more agents would want to buy and less sell and the return would adjust downwards, would it be lower, less agents would want to buy and more sell.

The main insight when comparing these models was that in the model, in which the discount factor is heterogeneous, there is a positive relationship between turnover and expectations. In the model, in which there are heterogeneous expectations about future returns, the same relationship is negative. Equation 8 gives clues to why this result emerges. In the first model, if we were to measure a higher expected return in the data, it would imply (as agents are perfectly rational) that the average discount factor has decreased. In other words: agents value the future less and are thus demanding higher expected returns to hold the asset. I assume that the distribution around the average discount factor β_t does not change over time, and we know that $\bar{\beta} = (1 + \kappa)\,\beta_t$ and therefore the thresholds are moving closer to the average discount factor and the no-trade zone shrinks.

If however the source of heterogeneity are agents' expectations about the asset's future value, the opposite effect occurs. An increase in measured return expectations implies that the average discount factor is lower too, but trade is governed by the distribution of individual expectations, not the distribution of the discount factor. In the distribution of return expectations $f_{\frac{\mathcal{E}}{q}}$, if the mean $\mathcal{E}_t$ increases, the thresholds move further apart and the no-trade zone increases, as $\bar{\mathcal{E}}_t = (1 + \kappa)\,\mathcal{E}_t$. Thus, we see less turnover in periods with high return expectations.

Another difference worth highlighting is the interpretation of return expectations between the models. In the model with heterogeneous discount factors, the agents are perfectly rational, but they differ in their preferences and thus demand different minimum expected returns to hold the assets. All agents agree that the expected return in the market is $E_t\left[R_{t+1}\right]$. If we would ask these agents (as return expectation surveys do) what return they expect, there would not be a large dispersion. This model of the world lends itself

[5]It is well established in the asset pricing literature that one of the two must be time-varying to explain asset price movements.

to comparison with model-based expectation measures. In the model with heterogeneous expectations, however, agents are not rational. They agree-to-disagree about what the expected return next period will be. In such a model the expected return is determined again by the discount factor, and thus by the preferences of agents. However, agents do not understand this and stick with their subjective expectations. This model set-up lends itself to be mapped to survey-based expectations measures, in which respondents clearly agree-to-disagree and display a wide dispersion of opinions.

3 Empirical Evidence

In the simple model of Section 2, we find that there is a positive relationship between expected returns and turnover when market participants are heterogeneous in their discount factors, and a negative relationship when market participants are heterogeneous in their return expectations. In the following empirical section, I describe the empirical correlational relationship of turnover and different measures of expectations, both model- and survey-based. I focus on simple correlations, as without an assumption on the functional form of the distributions f_β and $f_{\frac{\varepsilon}{q}}$, we do not know the functional form of turnover as a function of return expectations either. It could be non-linear, but we at least know that the model predicts the relationship to be monotonic. Thus, focusing on the simple linear correlation is a reasonable approach to understand the relationship between the variables. I will restrict my analysis to the US stock and housing markets, due to reasons of data availability and quality.

3.1 Data

I will use three types of data in the analysis, for both the US housing and the US stock market, respectively: data on aggregate turnover, model-based expected return measures (meaning measures, which according to a model with rational agents approximate expected returns), and survey-based measures, for which individuals (usually investors) are asked about their expectations regarding the developments in the respective markets in the near future (usually over the next year). In the following, I will describe my data sources in detail.

3.1.1 Stock Market

Turnover

I use two different series for turnover in the stock market. The first series is value weighted turnover for the US stock market, as calculated from the CRSP data set.[6]. This data set begins as early as 1925, but for reliability reasons I use the time series starting in 1962, as do Lo and Wang (2010). The data set is available on a daily basis, but for the analysis here the highest frequency needed is monthly.
The second data set is the turnover of the 1000 largest shares traded on the New York Stock Exchange, which

[6]See the Lo and Wang (2010) handbook chapter for the detailed calculation.

begins in 1973, and is again available daily, but aggregated to monthly for my purposes.[7] Both measures are highly correlated and in the following I report results from using either one or the other, not both.

Model-based Return Measures

I use three different types of model-based return measures: the realized market returns from both the CRSP (value-weighted) and NYSE data sets (described above for turnover), the value-weighted dividend-price ratio from CRSP, building on the widely accepted idea that fluctuations in the dividend-price ratio are reflecting fluctuations in return expectations, much more so than future dividend growth, see Cochrane (2011), and the consumption-wealth ration, as described in Lettau and Ludvigson (2001).[8] The rationale behind the consumption-wealth ratio is that if the permanent income hypothesis holds, high prices that are determined by low required returns should imply a lower consumption-wealth ratio. In the model in this paper, in which dividends are consumed on the spot, this idea is directly related to the dividend-price ratio.

While I am interested in the correlation of turnover at time t with the dividend-price ratio and the consumption-wealth ration at time t, the relevant correlation of realized returns and turnover is that of turnover in t and realized returns over the period between t and $t + s$, as under rational expectations realized returns today are an approximation for expected returns in the present. Throughout, I will calculate the correlation for $s = 12$ months ahead.

Survey-based return measures

There are several surveys available that ask participants about their expectations for the future development of the stock market. I use the following surveys: the UBS/Gallup survey, the Graham-Harvey/Duke Fuqua CFO survey[9], the Livingston Survey[10], the survey administered by Robert Shiller[11], and finally the weekly "Investor Sentiment" survey by the American Association of Individual Investors.[12] In Table X the five surveys are described in detail. While some of the measures directly map to expectations of expected returns, others, like the AAII survey, instead ask investors whether they think the market will go up, down, or remain the same. In this case I use the difference between the share of bulls and bears as an approximation for the return. This makes sense for two reasons. Firstly, Greenwood and Shleifer (2014) show that these qualitative measures are highly correlated to survey measures that ask directly for expected returns. Secondly, it maps to the theory: in the second model above there is a monotone relationship between the ratio of bulls and bears and the average expected return (as the distribution of individual expected returns is fixed and symmetric

[7] These data are available from Refinitiv Datastream as "US-DS Market".
[8] These data are available on Martin Lettau's website.
[9] Made available on the website of the Duke CFO Global Business Outlook.
[10] Administered and made available by the Federal Reserve Bank of Philadelphia.
[11] Made available at the Yale International Center for Finance as "United States Stock Market Confidence Indices".
[12] Made available on the AAII website.

around the average expected return).

3.1.2 Housing Market

Turnover

For the US housing market, I calculate turnover as the ratio of sales of existing homes relative to the total housing stock. The number of sales for existing homes is collected and made available by the National Association of Realtors, and is reported on a monthly basis since 1970. I take the US housing stock from the American Housing Survey, administered by the Department of Housing and Urban Development and the U.S. Census Bureau.

Model-based Expectation

I calculate the return on housing, using the House Price Index provided by the Federal Housing Finance Agency, which starts in 1970. To calculate the return, rent is treated like a dividend and derived from the Rent of Primary Residence CPI, which is calculated by the Bureau of Labor Statistics. If I just used home price growth instead of treating the average rent payment like a dividend, I would get very similar results, as most of the movement in housing returns is driven by price changes.

Survey-based Expectation Measures

I use the Zillow Home Price Expectation survey, which was initiated in 2010 and is administered on a quarterly basis. Investors and Experts are asked what they expect for the growth of real estate prices year-over-year. As the survey asks for the total change during the calendar year (so not the year-on-year change), I use the expectations that participants report to have for the year-over-year change in the next calendar year.

3.2 Empirical Relationship of Asset Turnover and Survey-based Expectations

For both the stock and housing markets, I linearly detrend turnover and compute the correlation coefficient with each of the respective survey-based return expectation series. The results are summarized in Table 1. It seems clear that the survey-based expectation measures do not provide much supportive evidence in either direction: for the Shiller, Livingston and AAII measures the correlation coefficient is weakly positive (in the case of Shiller's valuation measure, even moderately positive). However, in the Gallup and Duke CFO surveys the relationship is moderately negative. Furthermore, in the Duke and Livingston surveys the respective relationships are not statistically significant. The co-movement of the series with turnover are displayed in Figures 1–10 in the appendix.

Admittedly, the surveys all differ slightly, or even substantially, in their methodology, so different results on the sign could be driven by the differences in sample selection and by the fact that with some surveys, I am restricted to a qualitative measure, such as the share of bullish investors, while with others I use a quantitative return expectation. However, it is striking that especially those surveys that are similar in methodology, like AAII and Gallup, don't seem to be statistically related to turnover in the same way. The most immediate interpretation is that there is simply no robust relationship between survey-expectation and turnover.

In the housing market, we observe a significant positive correlation of 50% between survey-expectations from the Zillow survey and housing turnover. Of course for this example the sample period is relatively short, as the Zillow survey was only initiated in 2010 and during the sample period home sales and prices have done nothing but increase steadily. The corresponding figure is Figure 11. [13]

3.3 Empirical Relationship of Asset Turnover and Model-based Expectations

Table 2 summarizes the correlation of stock market turnover with the model-based expectation measures. Again, the evidence for any side is weak: we see no correlation for the value-based realized return over the subsequent 12 months after turnover is measured, only a small positive correlation for the dividend-price ratio, and a negative, but insignificant, correlation for the consumption-wealth ratio. As with the survey-based measures, there is no smoking gun either way. Figures 12–14 display the co-movement of the respective series. As a robustness check, I also tested the correlation while excluding the years 2006-2012 (inclusive) from the sample, which displayed abnormally high turnover, but without a significant change in the results.

In the housing market we again observe a moderately positive correlation between the market return and housing turnover, as can be seen in Figure 15. Measured at annual frequency, the correlation with the contemporaneous realized return is 43%. If I move to the quarterly frequency, measure turnover per quarter and compute the correlation with the realized return in the subsequent four quarters, we find a correlation coefficient of 44%. In summary, I conclude that the evidence for any kind of significant relationship of turnover and measures of return expectations in the US stock market is weak, and that, if anything, there is evidence for a moderately positive relationship in the US housing market.

4 Conclusion

I presented a simple model of heterogeneous trading and showed that different sources of heterogeneity imply different signs for the relationship between trading activity (measured as asset turnover) and return expectations. The model assumes a fixed distribution of the heterogeneous variable around its mean and

[13] An interesting artifact is that despite decent growth of home prices, survey participants are not further increasing their return expectations (as we might expect from extrapolators, which are a common building block in many behavioral models of housing markets). They seem to have settled in at around an expected growth of 3-4% annual nominal increase.

Table 1: Correlation of Detrended Stock Market Turnover with Survey-based Expectation Measures

Survey-based Measures	Correlation	Period + Frequency
Shiller One-Year Individual	0.2*** [0.07,0.32]	Biannually from Apr 1999, monthly since Jul 2001.
Shiller One-Year Institutional	0.27*** [0.15,0.38]	Biannually from Oct 1989, monthly since Jul 2001.
Shiller Valuation Individual	0.53*** [0.43, 0.62]	Biannually from Apr 1999, monthly since Jul 2001.
Shiller Valuation Institutional	0.43*** [0.33,0.53]	Biannually from Oct 1989, monthly since Jul 2001.
AAII Member Survey	0.34*** [0.26,0.43]	Weekly, aggregated to monthly Jun 1987- Nov 2020.
Gallup/UBS Survey	-0.59*** [-0.68,-0.48]	Monthly (with gaps) Oct 1996 - Nov 2011
Duke CFO Mean	-0.14 [-0.36, 0.1]	Quarterly from Oct 2000 - Dec 2020
Duke CFO Median	-0.3** [-0.51, -0.06]	Quarterly from Mar 2004 - Dec 2020
Livingston Mean	0.12 [-0.09, 0.31]	Biannually from Jun 1973 - Jun 2020
Livingston Median	0.33*** [0.14, 0.5]	Biannually from Jun 1973 - Jun 2020
	p-value:<1%***, <5%**, <10%*	All Shiller Survey Data up to Nov 2020

proportional transaction costs. These ingredients imply that an increase in the mean of the heterogeneous variable decreases the no-trade region and reduces turnover. If the source of heterogeneity is the discount

Table 2: Correlation of Detrended Stock Market Turnover with Model-based Expectation Measures

Model-based Measures	Correlation	Period + Frequency
12 Months Realized Returns	-0.06	Monthly from 1962-2020.
	[-0.13, 0.02]	
Dividend-Price Ratio	0.23***	Monthly from 1962-2020.
	[0.16, 0.3]	
Consumption-Wealth Ratio	-0.11	Quarterly from Apr 1952.
	[-0.22, 0.02]	

p-value:<1%***, <5%**, <10%*

factor of traders, the model predicts a positive relationship between turnover and expected returns, which is driven by the fact that a low average discount factor (which comes with high turnover) implies that the individually required returns to hold the asset, and therefore the market expected return, are high.

However, if the source of heterogeneity in the model is that traders have different expectations of future returns, the relationship between turnover and expected returns is predicted to be negative! The reason is that if average expected returns increase in this version of the model, the no-trade zone widens and turnover decreases.

Empirically, I set out to analyze the relationship between turnover and expected returns in the data, using both model-based and survey-based measures. The evidence on the sign, however, is mixed, with the correlation alternating between negative and positive, depending on the exact measure and the sample period. If anything, there is a moderately positive relationship between turnover and return expectations in the housing market. This result might seem disappointing, but it is in fact interesting, as it provides evidence against the view that trading frenzies are a symptom of asset price bubbles or "exuberant" expectations.

There are avenues for future research: a general model of asset prices and turnover is needed to think about how the additional information added by turnover could help researchers to disentangle the stochastic discount factor, irrational exuberance, and rational bubbles as drivers of volatility in asset markets. Intuitively, the model tells us that if model-based measures of returns are high, but turnover is low, the reason behind the high expected returns cannot be the stochastic discount factor, but a richer model becomes necessary if we want to think about these phenomena more thoroughly when agents are risk-averse.

Lastly, future research can explore the properties of the distribution of survey expectations. In the model, I assumed that the distribution of expectations only changes over time in its first moment. However, this needs to be tested and if there are significant changes in the distribution over time, this could again be exploited to improve our understanding of price volatility in asset markets.

5 Figures

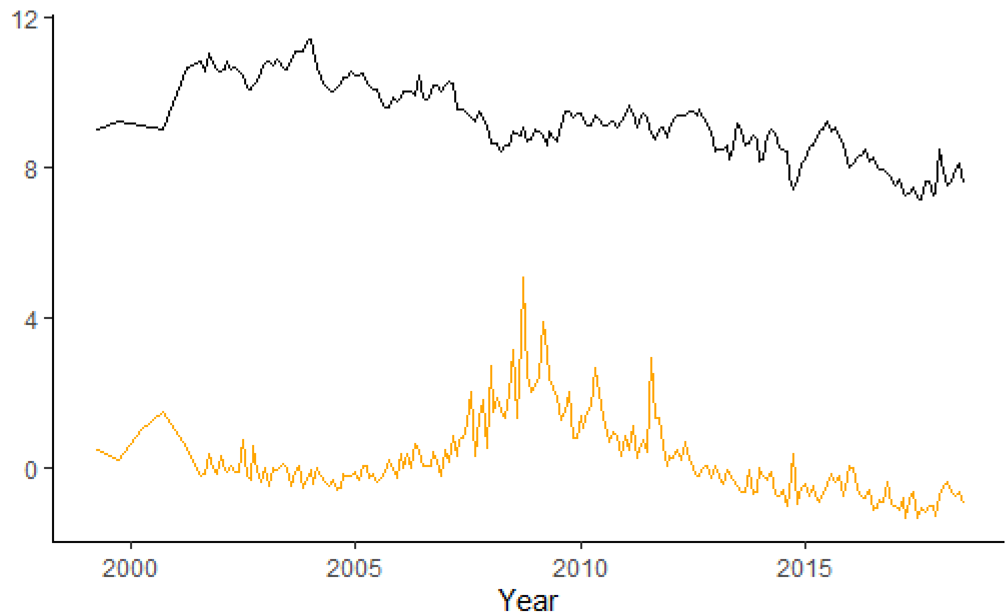

Figure 1: Shiller Individual Investors One-year Ahead Expectations.

Displayed are the time series depicting the linearly detrended turnover in the US and the time series depicting %bulls-%bears in the Shiller Individual Investor Survey, in which investors are predicting the market return one year ahead. Both series are standardized by dividing through their respective standard deviations. The correlation coefficient is 20%.

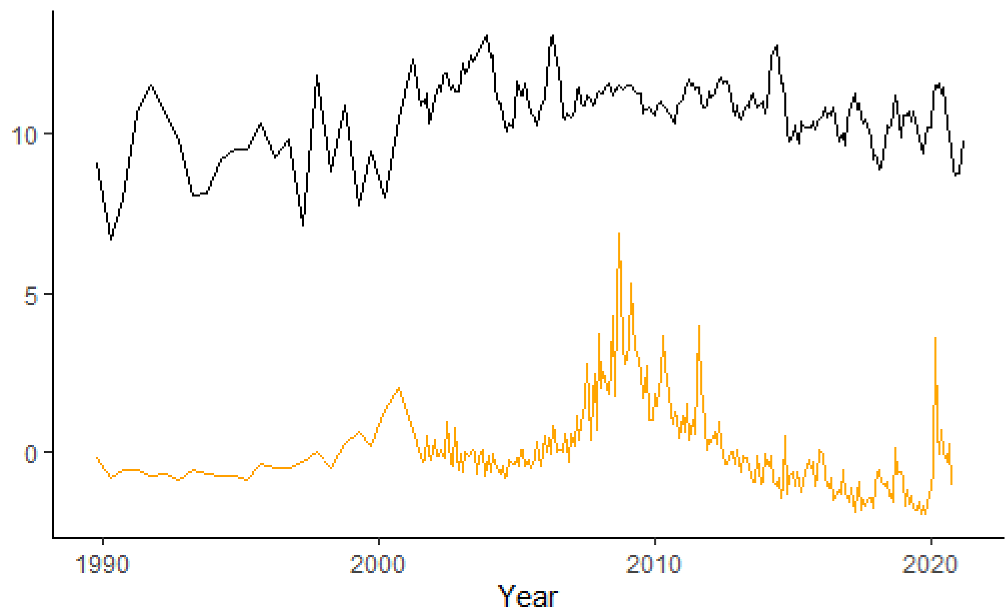

Figure 2: Shiller Institutional Investors One-year Ahead Expectations.

Displayed are the time series depicting the linearly detrended turnover in the US stock market (orange) and the time series depicting %bulls-%bears in the Shiller Institutional Investor Survey (black), in which investors are predicting the market return one year ahead. Both series are standardized by dividing through their respective standard deviations. The correlation coefficient is 27%.

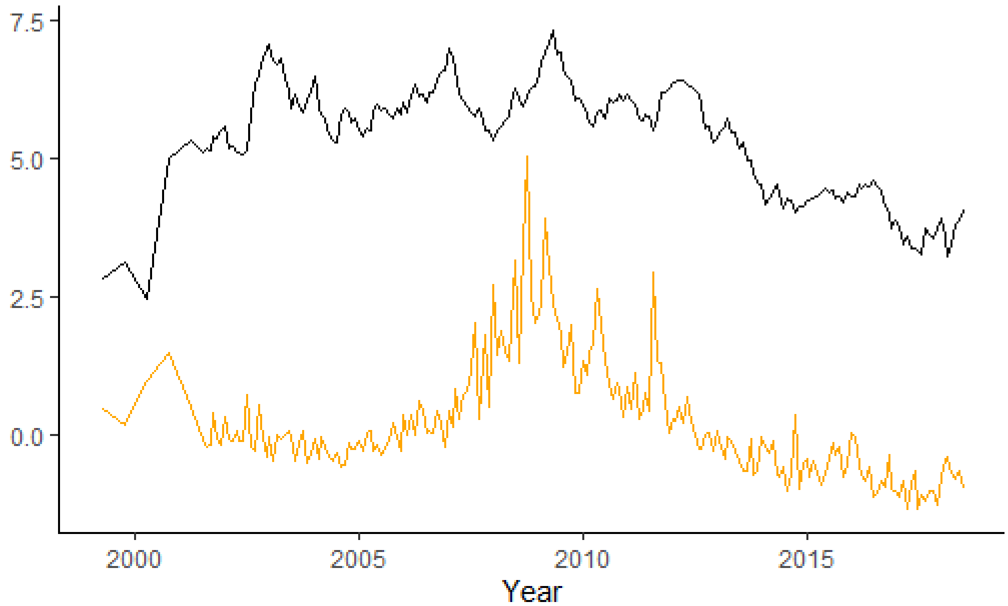

Figure 3: Shiller Individual Investors Market Valuation Survey.

Displayed are the time series depicting the linearly detrended turnover in the US stock market (orange) and the time series depicting %bulls-%bears in the Shiller Individual Investor Survey (black), in which investors are asked whether they perceive the market to be under- or overvalued. Both series are standardized by dividing through their respective standard deviations. The correlation coefficient is 53%.

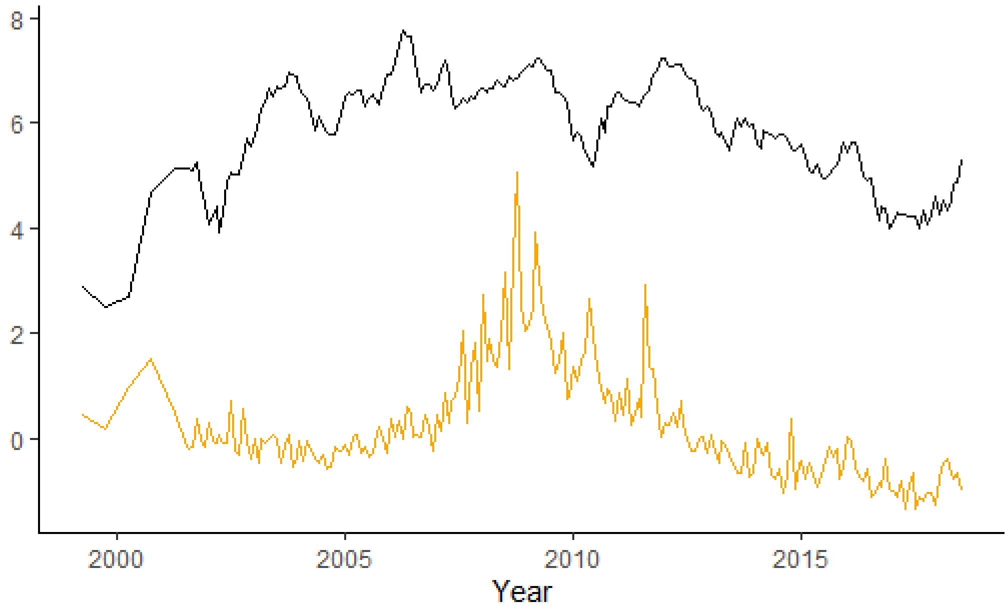

Figure 4: Shiller Individual Investors Market Valuation Survey.

Displayed are the time series depicting the linearly detrended turnover in the US stock market (orange) and the time series depicting %bulls-%bears in the Shiller Institutional Investor Survey (black), in which investors are asked whether they perceive the market to be under- or overvalued. Both series are standardized by dividing through their respective standard deviations. The correlation coefficient is 43%.

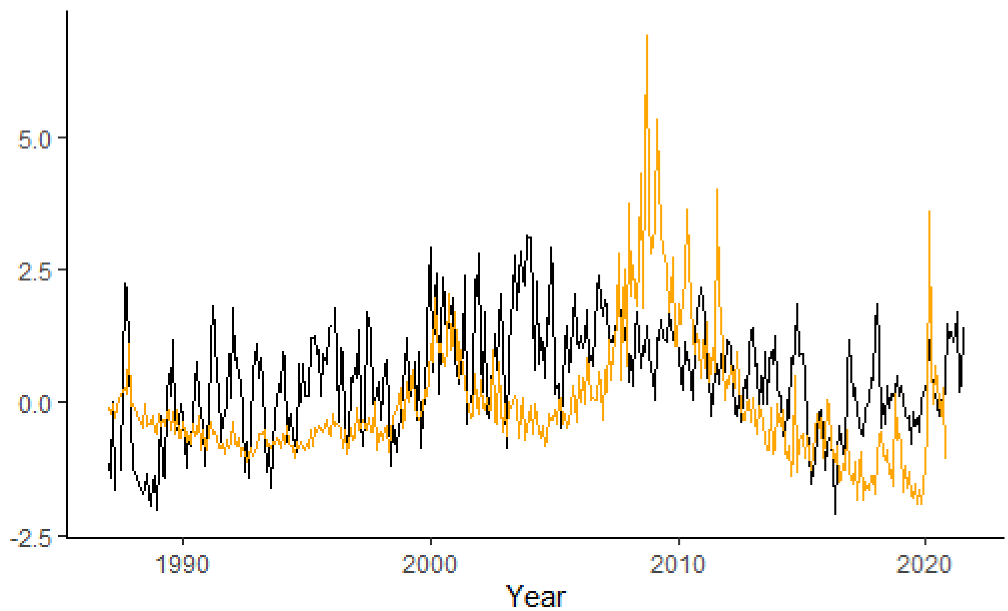

Figure 5: American Association of Individual Investors Member Survey.

Displayed are the time series depicting the linearly detrended turnover in the US stock market (orange) and the time series depicting %bulls-%bears in the American Association of Individual Investors Member Survey (black), in which AAII members are asked whether they are "bullish" or "bearish". Both series are standardized by dividing through their respective standard deviations. The correlation coefficient is 34%.

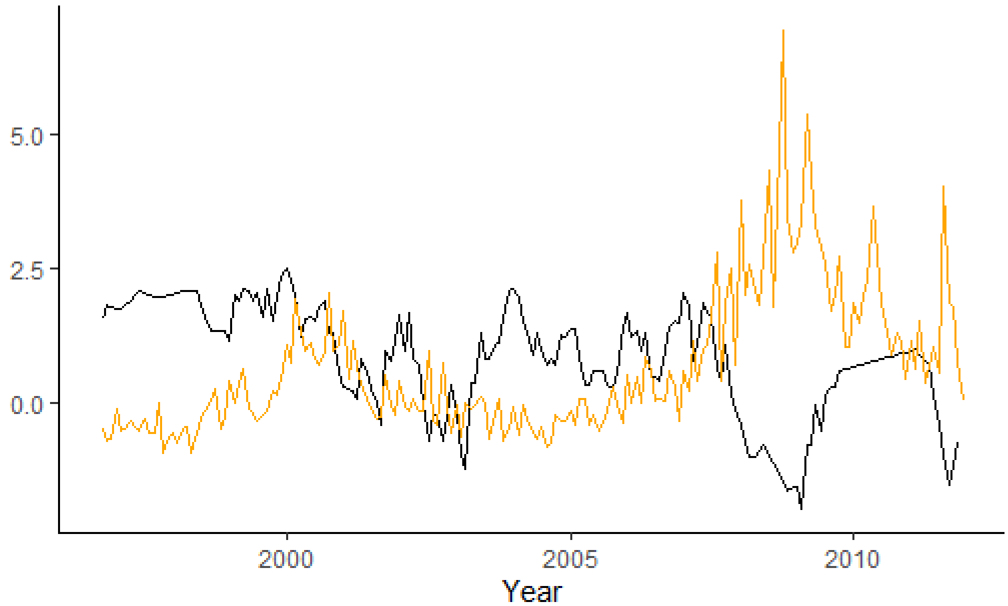

Figure 6: Gallup/UBS Investor Survey.

Displayed are the time series depicting the linearly detrended turnover in the US stock market (orange) and the time series depicting %bulls-%bears in the Gallup/UBS Investor Survey (black), in which investors are asked whether they are "optimistic" or "pessimistic" with regard to market returns. Both series are standardized by dividing through their respective standard deviations. The correlation coefficient is -59%.

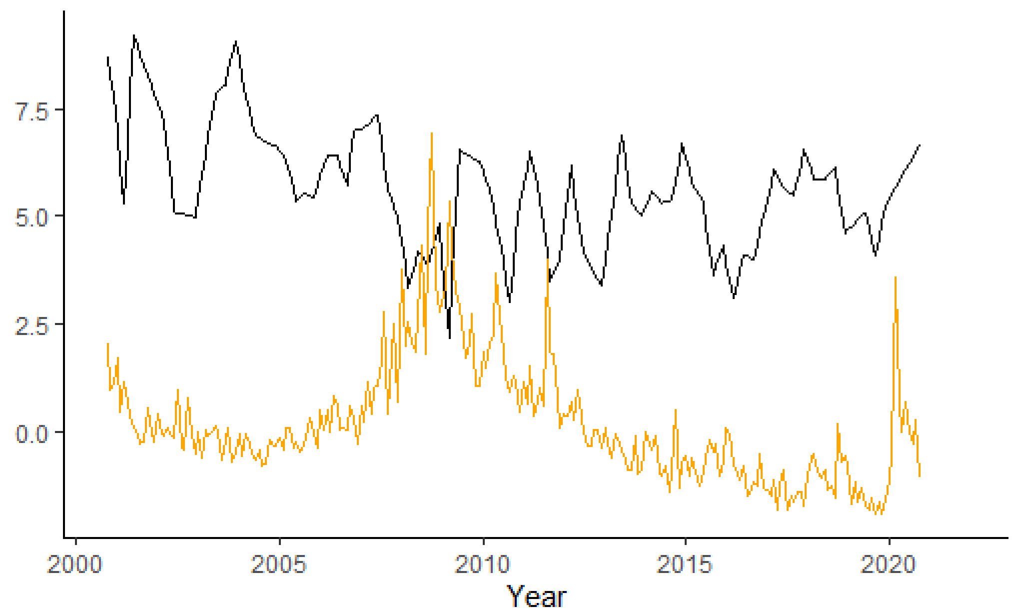

Figure 7: Duke CFO Survey: Mean One-Year Expectation for the S&P500 Return.

Displayed are the time series depicting the linearly detrended turnover in the US stock market (orange) and the time series depicting the average return expectation in the Duke CFO Survey (black), in which CFOs are asked about what they expect the return of the S&P 500 to be for the next year. Both series are standardized by dividing through their respective standard deviations. The correlation coefficient is -14%.

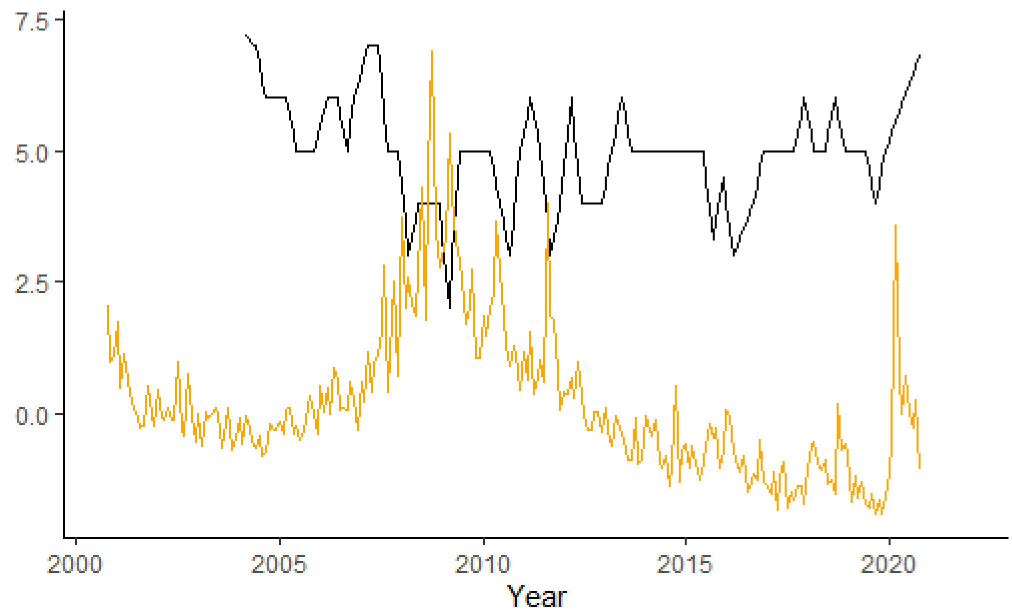

Figure 8: Duke CFO Survey: Median One-Year Expectation for the S&P500 Return.

Displayed are the time series depicting the linearly detrended turnover in the US stock market (orange) and the time series depicting the median return expectation in the Duke CFO Survey (black), in which CFOs are asked about what they expect the return of the S&P 500 to be for the next year. Both series are standardized by dividing through their respective standard deviations. The correlation coefficient is -30%.

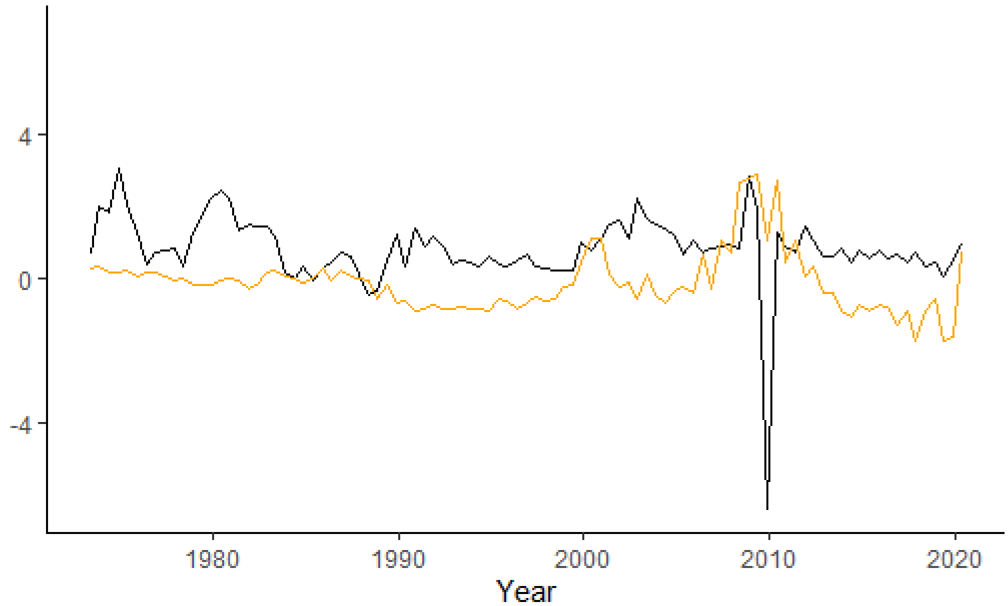

Figure 9: Livingston Survey, Mean Half-year Expectation for the S&P500 Return.

Displayed are the time series depicting the linearly detrended turnover in the US stock market (orange) and the time series depicting the mean return expectation in the Livingston Survey (black), in which economists are asked about what they expect the return of the S&P 500 to be for the next half-year. Both series are standardized by dividing through their respective standard deviations. The correlation coefficient is 12%.

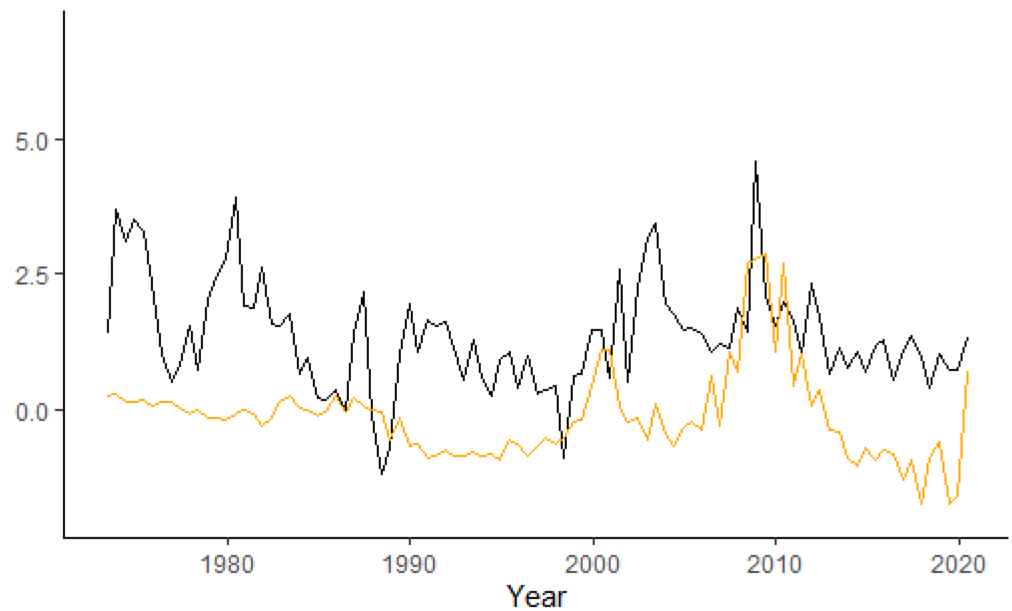

Figure 10: Livingston Survey, Median Half-year Expectation for the S&P500 Return.

Displayed are the time series depicting the linearly detrended turnover in the US stock market (orange) and the time series depicting the median return expectation in the Livingston Survey (black), in which economists are asked about what they expect the return of the S&P 500 to be for the next half-year. Both series are standardized by dividing through their respective standard deviations. The correlation coefficient is 33%.

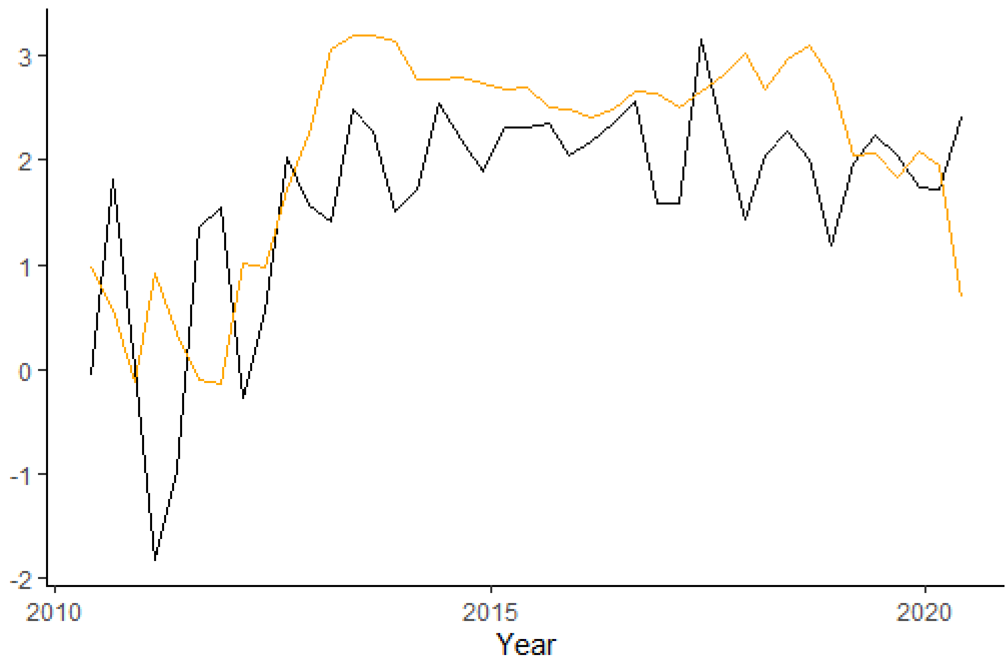

Figure 11: Zillow Year-over-Year Expectations (Black) and Home Turnover (Orange).

Displayed are the time series depicting the linearly detrended turnover in the US housing market (orange) and the time series depicting the average return expectation in the Zillow Home Price Expectation Survey (black), in which home owners are asked about what they expect the average price growth in the US housing market to be for the coming calendar year. Both series are standardized by dividing through their respective standard deviations. The correlation coefficient is 50%.

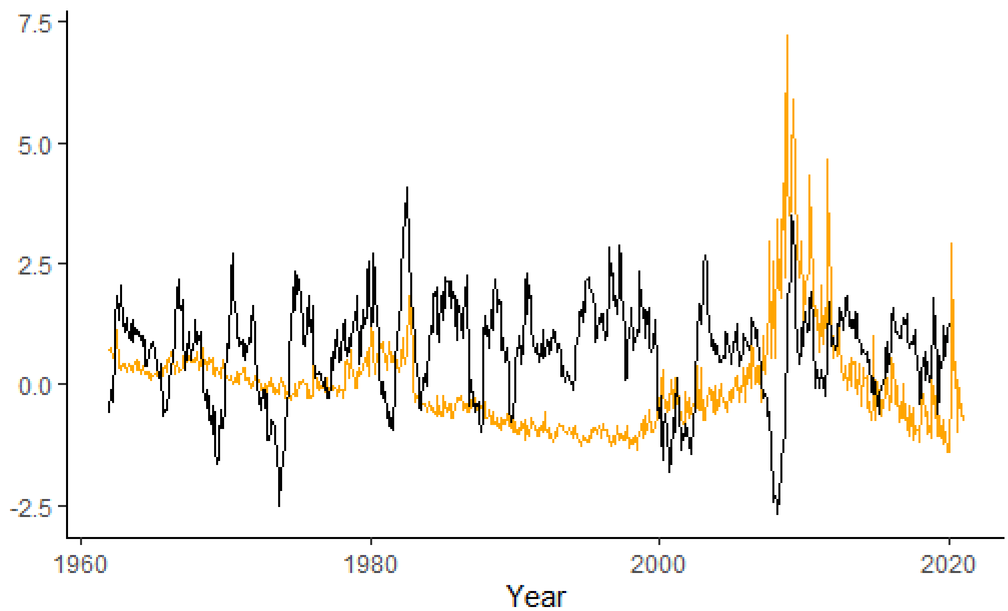

Figure 12: Subsequent Value-Weighted 12-Month Market Return on US Stocks.

Displayed are the time series depicting the linearly detrended turnover in the US stock market (orange) and the time series depicting the return in the 12 months after the turnover was measured (black). Value-weighted data from CRSP. Both series are standardized by dividing through their respective standard deviations. The correlation coefficient is 6%.

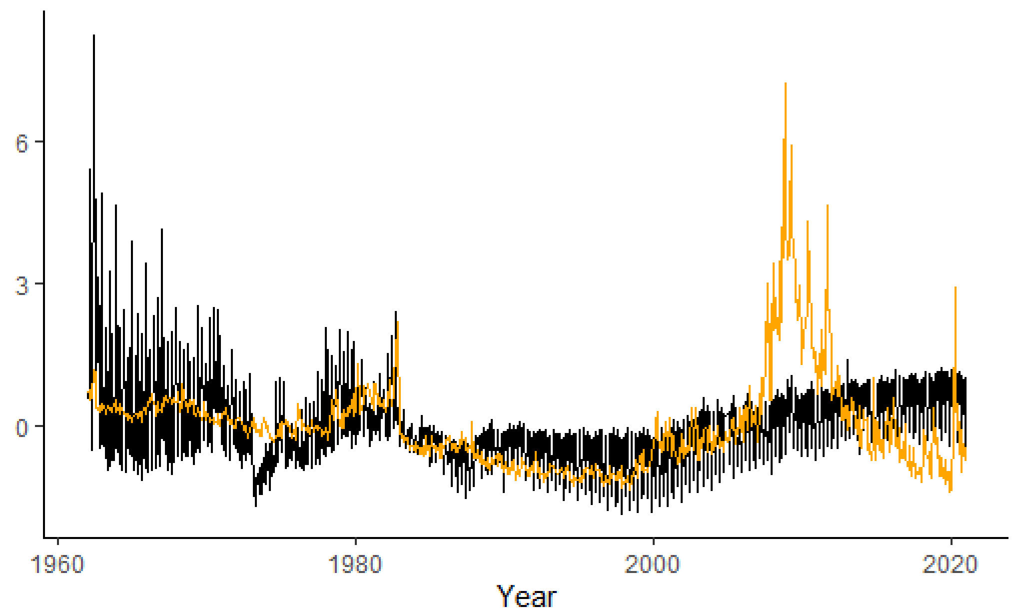

Figure 13: Value-Weighted Dividend-Price Ratio of US Stocks.

Displayed are the time series depicting the linearly detrended turnover in the US stock market (orange) and the time series depicting the dividend-price ratio (black). Value-weighted data from CRSP. Both series are standardized by dividing through their respective standard deviations. The correlation coefficient is 23%.

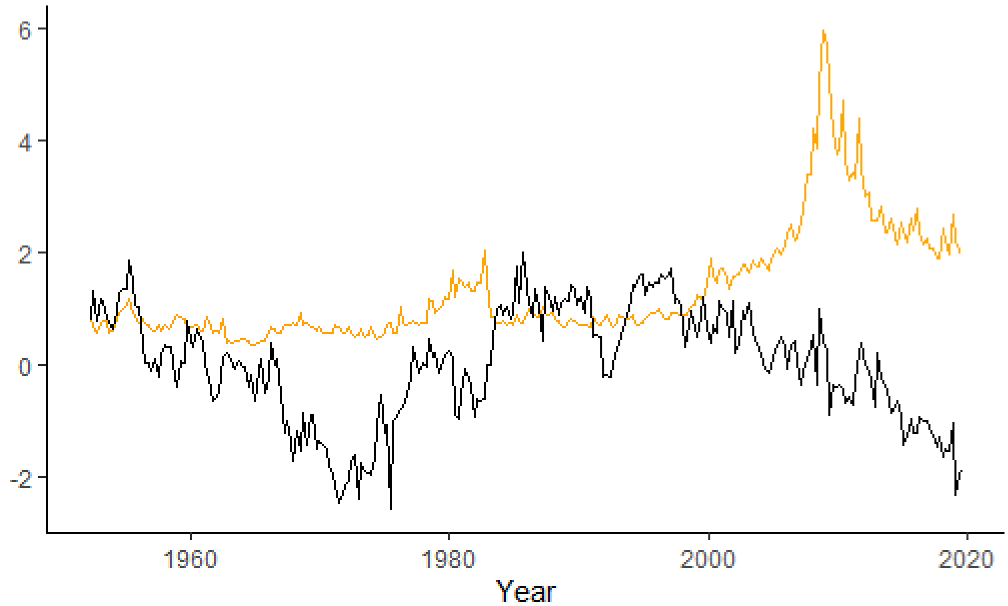

Figure 14: Consumption-Wealth Ratio.

Displayed are the time series depicting the linearly detrended turnover in the US stock market (orange) and the time series depicting the consumption-wealth ratio (black). Both series are standardized by dividing through their respective standard deviations. The correlation coefficient is -10%.

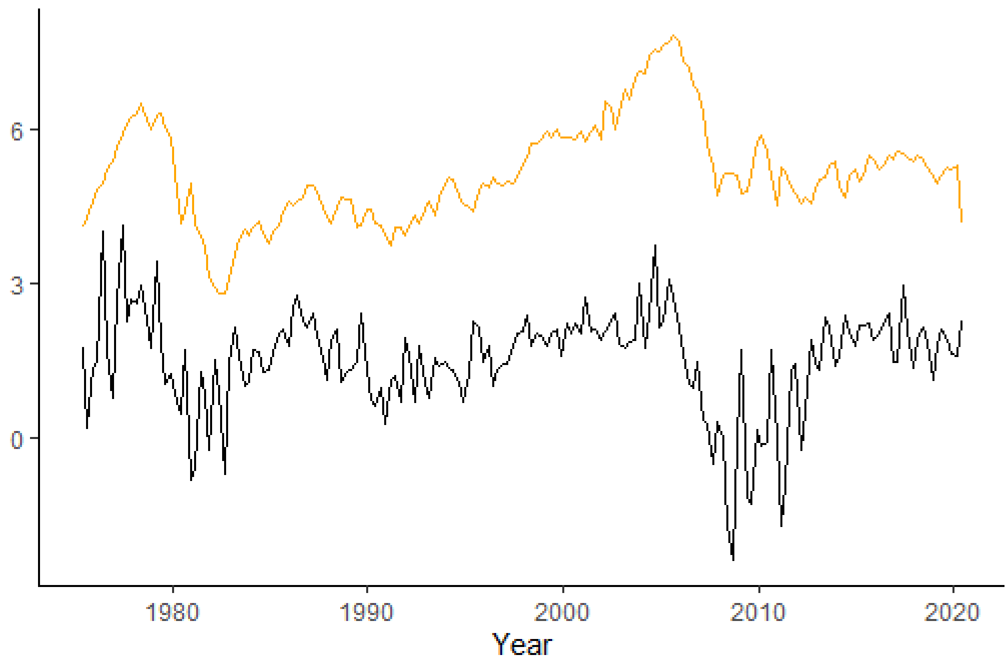

Figure 15: Market Return on Housing.

Displayed are the time series depicting the linearly detrended turnover in the US housing market (orange) and the time series depicting the market return on housing as computed from the FHFA index and Rent CPI (black). Both series are standardized by dividing through their respective standard deviations. The correlation coefficient is 43% at annual frequency. If we are calculating the returns over the subsequent four quarters at quarterly frequency, we find a correlation coefficient of 44%.

6 Tables

Table 3: Expectation Surveys: Overview

Survey-based Measures	Description
Shiller Individual	Biannual from Apr 1999, monthly Jul 2001-Nov 2020. A random sample of high-income US Americans reports their expectations for the US stock market, including on by how much they think the Dow Jones will have increased in a year and whether they think the market is under- or overvalued. The measures used in the empirical analysis is the share of the sample who expect the Dow Jones Index to increase (One-year) and the share of the sample who think the market is undervalued (Valuation). Sample size is around 100 on average.
Shiller Institutional	Biannual from Oct 1989, monthly Jul 2001-Nov 2020. A random sample of US-based institutional investors is asked the same set of questions as in the "Shiller Individual" sample. Sample size is around 100 on average.
AAII Member Survey	Weekly, aggregated to monthly. Jun 1987- Nov 2020. Conducted via the AAII main publication (today the website). Respondents can choose whether they are "Bullish", "Neutral", or "Bearish". The measure used in the empirical analysis is %bulls-%bears.

Gallup/UBS Survey	Monthly (with notable gaps). Oct 1996 - Nov 2011. Telephone Interviews with 500-1000 participants who have invested >$10000 in the market. They are asked about their level of optimism about the stock market. The measure used in the empirical analysis is %optimists-%pessimists.
Duke CFO Survey	Quarterly from Oct 2000 - Dec 2020. A stable panel of CFOs from a diverse set of companies is asked how they think the return of the S&P 500 will be over the next year.
Livingston Survey	Biannual from 1952 - Jun 2020. Participants are economists in academia, industry, governments, and institutions. They are asked to forecast the level of the S&P 500 "in six months" and "in twelve months". As I can't identify when the response was given, and thus an expected return cannot be inferred, I use the growth of the estimate for the level "in six months" and "in twelve months" as a return expectation for months six to twelve.

Table 4: Correlation of Non-Detrended Stock Market Turnover with Survey-based Expectation Measures

Survey-based Measures	Correlation	Period + Frequency
Shiller One-Year Individual	-0.18*** [-0.3, -0.05]	Biannually from Apr 1999, monthly since Jul 2001.
Shiller One-Year Institutional	0.26*** [0.14, 0.37]	Biannually from Oct 1989, monthly since Jul 2001.
Shiller Valuation Individual	0.26*** [0.14, 0.37]	Biannually from Apr 1999, monthly since Jul 2001.
Shiller Valuation Institutional	0.36*** [0.25,0.46]	Biannually from Oct 1989, monthly since Jul 2001.
AAII Member Survey	0.32*** [0.23 0.41]	Weekly, aggregated to monthly Jun 1987- Nov 2020.
Gallup/UBS Survey	-0.64*** [-0.72 -0.55]	Monthly (with gaps) Oct 1996 - Nov 2011
Duke CFO Mean	-0.38*** [-0.56 -0.16]	Quarterly from Oct 2000 - Dec 2020
Duke CFO Median	-0.48** [-0.63 -0.23]	Quarterly from Mar 2004 - Dec 2020
Livingston Mean	-0.12 [-0.31 0.085]	Biannually from Jun 1973 - Jun 2020
Livingston Median	0.015 [-0.19 0.22]	Biannually from Jun 1973 - Jun 2020
	p-value:<1%***, <5%**, <10%*	All Shiller Survey Data up to Nov 2020

7 Proofs

7..1 Proposition 1

Proof. Under the i.i.d assumption on β_t^i: $\bar{\text{p}}_t = P\left(\beta_t^i > \bar{\beta}_t\right)$ and $\underline{\text{p}}_t = P\left(\beta_t^i < \underline{\beta}_t\right)$.

From Equation 1 and $N = 1/2$ follows

$$P\left(\beta_t^i > \bar{\beta}_t\right) = P\left(\beta_t^i < \underline{\beta}_t\right).$$

Both the owners and the non-owners will follow trading policies conditional on their respective draws of β_t^i.

Non-owner i buys if

$$\beta_t^i E_t\left[V^o\left(\beta_{t+1}^i, q_{t+1}\right) - V^{no}\left(\beta_{t+1}^i, q_{t+1}\right)\right] > (1+\kappa)\, q_t, \tag{9}$$

Similarly, owner i sells if

$$(1-\kappa)\, q_t > \beta_t^i E_t\left[V^o\left(\beta_{t+1}^i, q_{t+1}\right) - V^{no}\left(\beta_{t+1}^i, q_{t+1}\right)\right]. \tag{10}$$

Equations 9 and 10 define the individual thresholds for the discount factor for buying and selling the asset:

$$(1+\kappa)\, q_t = \bar{\beta}_t E_t\left[G\left(\beta_{t+1}^i\right) | \beta_t^i = \bar{\beta}_t\right], \tag{11}$$

and

$$(1-\kappa)\, q_t = \underline{\beta}_t E_t\left[G\left(\beta_{t+1}^i\right) | \beta_t^i = \underline{\beta}_t\right], \tag{12}$$

where

$$G\left(\beta_{t+1}^i\right) \equiv V^o\left(\beta_{t+1}^i, q_{t+1}\right) - V^{no}\left(\beta_{t+1}^i, q_{t+1}\right).$$

Note that, due to the i.i.d assumption on β_t^i the expected option value of holding the asset in $t+1$ is independent of the individual discount factor in t, and thus:

$$E_t\left[G\left(\beta_{t+1}^i\right) | \beta_t^i = \bar{\beta}_t\right] = E_t\left[G\left(\beta_{t+1}^i\right) | \beta_t^i = \underline{\beta}_t\right] = E_t\left[G\left(\beta_{t+1}^i\right)\right].$$

From the threshold equations 11 and 12 follow (by respectively adding and subtracting the two equations):

an asset pricing equation, which is

$$q_t = \frac{\bar{\beta}_t + \underline{\beta}_t}{2} E_t\left[G\left(\beta_{t+1}^i\right)\right], \tag{13}$$

as well as an equation implicitly determining turnover, which is

$$\bar{\beta}_t - \underline{\beta}_t = \frac{2\kappa q_t}{E_t G\left(\beta^i_{t+1}\right)}. \tag{14}$$

The equilibrium price q_t, must be such that the number of owners in the selling region equals the number of owners in the buying region:

$$(1-N)\,\bar{\mathrm{p}}_t = N\underline{\mathrm{p}}_t.$$

Under the stated assumptions that $N = 1/2$ and f_β is symmetric around β_t:

$$\underline{\mathrm{p}}_t = \bar{\mathrm{p}}_t \implies \frac{\bar{\beta}_t + \underline{\beta}_t}{2} = \beta_t \implies q_t = \beta_t E_t\left[G\left(\beta^i_{t+1}\right)\right].$$

This result can be used to understand G:

$$E_t G\left(\beta^i_{t+1}\right) = E_t\left[V^o\left(\beta^i_{t+1}, q_{t+1}\right) - V^{no}\left(\beta^i_{t+1}, q_{t+1}\right)\right]$$

$$= E_t\{\bar{\mathrm{p}}_{t+1}\left(d_{t+1} + (1-\kappa)\,q_{t+1} + E^*_{t+1}\left[\beta^i_{t+1}|\beta^i_{t+1} < \underline{\beta}_{t+1}\right] E_{t+1}\left[V^{no}\left(\beta^i_{t+2}, q_{t+2}\right)\right]\right)$$

$$+\left(1-\bar{\mathrm{p}}_{t+1}\right)\left(d_{t+1} + E^*_{t+1}\left[\beta^i_{t+1}|\beta^i_{t+1} > \underline{\beta}_{t+1}\right] E_{t+1}\left[V^o\left(\beta^i_{t+2}, q_{t+2}\right)\right]\right)$$

$$-\bar{\mathrm{p}}_{t+1}\left(-(1+\kappa)\,q_{t+1} + E^*_{t+1}\left[\beta^i_{t+1}|\beta^i_{t+1} > \bar{\beta}_{t+1}\right] E_{t+1}\left[V^o\left(\beta^i_{t+2}, q_{t+2}\right)\right]\right)$$

$$-\left(1-\bar{\mathrm{p}}_{t+1}\right)\left(E^*_{t+1}\left[\beta^i_{t+1}|\beta^i_{t+1} < \bar{\beta}_{t+1}\right] E_{t+1}\left[V^{no}\left(\beta^i_{t+2}, q_{t+2}\right)\right]\right)\},$$

which we can summarize as: [14]

$$E_t G\left(\beta^i_{t+1}\right) = E_t\left[d_{t+1} + 2\bar{\mathrm{p}}_{t+1} q_{t+1} + \left(1-2\bar{\mathrm{p}}_{t+1}\right)\beta_{t+1} E_{t+1} G\left(\beta^i_{t+2}\right)\right].$$

By rolling forward we find that $E_t G$ is just the expected value of next period dividend and price:

$$q_{t+1} = \beta_{t+1} E_{t+1} G\left(\beta^i_{t+2}\right),$$

[14] Here, the operator E^*_{t+1} denotes the expected value for the individual discount factor, when the mean β_{t+1} and therefore the distribution f_β is known. I use $E^*_{t+1}\left[\beta^i_{t+1}|\beta^i_{t+1} > \bar{\beta}_{t+1}\right] - \beta_{t+1} = \beta_{t+1} - E^*_{t+1}\left[\beta^i_{t+1}|\beta^i_{t+1} < \underline{\beta}_{t+1}\right]$ and $E^*_{t+1}\left[\beta^i_{t+1}|\beta^i_{t+1} > \underline{\beta}_{t+1}\right] - \beta_{t+1} = \beta_{t+1} - E^*_{t+1}\left[\beta^i_{t+1}|\beta^i_{t+1} < \bar{\beta}_{t+1}\right]$, which again follows from the assumption of symmetry on f_β.

$$\implies \quad E_t G\left(\beta_{t+1}^i\right) = E_t\left[d_{t+1} + q_{t+1}\right] \ . \tag{15}$$

Thus, we find a standard asset pricing equation: $q_t = \beta_t E_t\left[d_{t+1} + q_{t+1}\right]$, which is Equation 2.

From Equation 14, we know that:

$$\bar{\beta}_t - \underline{\beta}_t = \frac{2\kappa q_t}{E_t\left[d_{t+1} + q_{t+1}\right]}.$$

As $P\left(\beta_t^i > \bar{\beta}_t\right) = P\left(\beta_t^i < \underline{\beta}_t\right)$, and the distribution f_β is symmetric around β_t, it is implied that $\bar{\beta}_t - \beta_t = \beta_t - \underline{\beta}_t$ or $\bar{\beta}_t - \underline{\beta}_t = 2\left(\bar{\beta}_t - \beta_t\right)$.

Thus, $\bar{\beta}_t - \beta_t = \kappa q_t / E_t\left[d_{t+1} + q_{t+1}\right]$ or in terms of returns $\bar{\beta}_t - \beta_t = \kappa / E_t\left[R_{t+1}\right]$. Turnover in this model is identical to the probability for an asset owner to sell, which is the mass of owners with individual discount factors β_t^i larger than $\beta_t + \kappa / E_t\left[R_{t+1}\right]$. Thus turnover is shown, as posited in Equation 3, to be:

$$\bar{p}_{t+1} = 1 - F_\beta\left(\frac{1+\kappa}{E_t\left[R_{t+1}\right]}\right).$$

□

7..2 Proposition 2

Proof. In this slightly different set-up, the value functions need to be adapted. The value for individual i of being an owner of an asset in t is

$$\begin{aligned} V_t^o\left(\beta_t, q_t\right) = \quad & \max\{d_t + (1-\kappa)\, q_t + \beta_t E_t^i\left[V^{no}\left(\beta_{t+1}, q_{t+1}\right)\right] \\ & d_t + \beta_t E_t^i\left[V^o\left(\beta_{t+1}, q_{t+1}\right)\right]\} \end{aligned},$$

while the value of not being an owner is accordingly

$$\begin{aligned} V_t^{no}\left(\beta_t, q_t\right) = \max\{ \quad & -(1+\kappa)\, q_t + \beta_t E_t^i\left[V^o\left(\beta_{t+1}, q_{t+1}\right)\right], \\ & \beta_t^i E_t^i\left[V^{no}\left(\beta_{t+1}, q_{t+1}\right)\right]\} \end{aligned}.$$

The thresholds for buying and selling, $\bar{E}_t$ and $\underline{E}_t$ are determined by

$$(1-\kappa)\, q_t = \beta_t \underline{\mathcal{E}}_t$$

and

$$(1+\kappa)\, q_t = \beta_t \bar{\mathcal{E}}_t.$$

These equations imply, following the same steps as in the proof of Proposition 1:

$$q_t = \frac{\beta_t}{2}\left(\bar{\mathcal{E}}_t + \underline{\mathcal{E}}_t\right) \tag{16}$$

and

$$\beta_t\left(\bar{\mathcal{E}}_t - \underline{\mathcal{E}}_t\right) = 2\kappa q_t. \tag{17}$$

For the market to clear, it must be true that

$$(1-N)\,P\left(\mathcal{E}_t^i > \bar{\mathcal{E}}_t\right) = NP\left(\mathcal{E}_t^i < \underline{\mathcal{E}}_t\right),$$

due to the i.i.d. assumptions on $\mathcal{E}_t^i/q_t$. With $N = 1/2$, it must be that $P\left(\mathcal{E}_t^i > \bar{\mathcal{E}}_t\right) \equiv \bar{p}_t = \underline{\mathrm{p}}_t \equiv P\left(\mathcal{E}_t^i < \underline{\mathcal{E}}_t\right)$. As the distribution $f_{\frac{\mathcal{E}}{q}}$ is symmetric, we know that

$$\frac{\bar{\mathcal{E}}_t + \underline{\mathcal{E}}_t}{2} = \mathcal{E}_t.$$

Furthermore,

$$E_t\left[V^o\left(\beta_{t+1}^i, q_{t+1}\right) - V^{no}\left(\beta_{t+1}^i, q_{t+1}\right)\right]$$

$$= E_t\{\bar{\mathrm{p}}_{t+1}\left(d_{t+1} + (1-\kappa)\,q_{t+1} + \beta_{t+1}E_{t+1}^*\left[E_{t+1}^i\left[V^{no}\left(\beta_{t+2}, q_{t+2}\right)\right]|\mathcal{E}_{t+1}^i < \underline{\mathcal{E}}_{t+1}\right]\right)$$

$$+\left(1-\bar{\mathrm{p}}_{t+1}\right)\left(d_{t+1} + \beta_{t+1}E_{t+1}^*\left[E_{t+1}^i\left[V^o\left(\beta_{t+2}, q_{t+2}\right)\right]|\mathcal{E}_{t+1}^i > \underline{\mathcal{E}}_{t+1}\right]\right)$$

$$-\bar{\mathrm{p}}_{t+1}\left(-(1+\kappa)\,q_{t+1} + \beta_{t+1}E_{t+1}^*\left[E_{t+1}^i\left[V^o\left(\beta_{t+2}, q_{t+2}\right)\right]|\mathcal{E}_{t+1}^i > \bar{\mathcal{E}}_{t+1}\right]\right)$$

$$-\left(1-\bar{\mathrm{p}}_{t+1}\right)\left(\beta_{t+1}E_{t+1}^*\left[E_{t+1}^i\left[V^{no}\left(\beta_{t+2}, q_{t+2}\right)\right]|\mathcal{E}_{t+1}^i < \bar{\mathcal{E}}_{t+1}\right]\right)\}.$$

We can summarize this expression as:[15]

$$E_t\left[V^o\left(\beta_{t+1}, q_{t+1}\right) - V^{no}\left(\beta_{t+1}, q_{t+1}\right)\right] = E_t\left[d_{t+1} + 2\bar{\mathrm{p}}_{t+1}q_{t+1} + \left(1-2\bar{\mathrm{p}}_{t+1}\right)\beta_{t+1}\mathcal{E}_{t+1}\right].$$

[15]Here, the operator E_{t+1}^* analogously to the previous proof, denotes the expected value for the individual future expectation, when the mean $\mathcal{E}_{t+1}$, and therefore the distribution $f_{\frac{\mathcal{E}}{q}}$, is known. I use $E_{t+1}^*\left[\mathcal{E}_{t+1}^i|\mathcal{E}_{t+1}^i > \bar{\mathcal{E}}_{t+1}\right] - \mathcal{E}_{t+1} = \mathcal{E}_{t+1} - E_{t+1}^*\left[\mathcal{E}_{t+1}^i|\mathcal{E}_{t+1}^i < \underline{\mathcal{E}}_{t+1}\right]$ and $E_{t+1}^*\left[\mathcal{E}_{t+1}^i|\mathcal{E}_{t+1}^i > \underline{\mathcal{E}}_{t+1}\right] - \mathcal{E}_{t+1} = \mathcal{E}_{t+1} - E_{t+1}^*\left[\mathcal{E}_{t+1}^i|\mathcal{E}_{t+1}^i < \bar{\mathcal{E}}_{t+1}\right]$, which again follows from the assumption of symmetry on $f_{\frac{\mathcal{E}}{q}}$.

By rolling forward we find that $E_t G$ is just the expected value of next period dividend and price:

$$q_{t+1} = \beta_{t+1}\mathcal{E}_{t+1},$$

$$\implies \quad E_t\left[V^o\left(\beta^i_{t+1}, q_{t+1}\right) - V^{no}\left(\beta^i_{t+1}, q_{t+1}\right)\right] = E_t\left[d_{t+1} + q_{t+1}\right] \; = \mathcal{E}_t. \tag{18}$$

Thus, we again find a standard asset pricing equation

$$q_t = \beta_t E_t\left[d_{t+1} + q_{t+1}\right].$$

Plugging the asset pricing equation into Equation 17, we find

$$\beta_t\left(\bar{\mathcal{E}}_t - \underline{\mathcal{E}}_t\right) = 2\kappa\beta_t\mathcal{E}_t.$$

Due to symmetry:

$$\bar{\mathcal{E}}_t - \mathcal{E}_t = \mathcal{E}_t - \underline{\mathcal{E}}_t,$$

which implies

$$\bar{\mathcal{E}}_t = (1+\kappa)\,\mathcal{E}_t.$$

Turnover is

$$\bar{p}_t = P\left(\mathcal{E}^i_t > \bar{\mathcal{E}}_t\right) P\left(\frac{\mathcal{E}^i_t}{q_t} > \frac{\bar{\mathcal{E}}_t}{q_t}\right) = 1 - F_{\frac{\mathcal{E}}{q}}\left((1+\kappa)\,\frac{E_t\left[d_{t+1} + q_{t+1}\right]}{q_t}\right)$$

$$= 1 - F_{\frac{\mathcal{E}}{q}}\left((1+\kappa)\,E_t\left[R_{t+1}\right]\right).$$

$\square$